The Geometry of Constant Function Market Makers

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Outline

Automated market makers

- Some contract with n assets
- Given a trade $\Delta \in \mathbf{R}^n$ accepts or rejects it via some rule
- Usually has some *reserves* $R \in \mathbf{R}^n_+$ from which it pays out
- Reserves are the updated $R \to R + \Delta$

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- Possibly the most famous example is Uniswap (v1/v2)

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- Uniswap, Balancer, Curve, Uniswap v3, Pendle, etc.
- (Essentially) All are *constant function market makers*
- Average over $\sim$ 1T USD in volume per year!

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- Most derived from an ‘algebraic view’
- But, fundamentally, CFMMs are geometric objects
- (With very intuitive properties)
- Can we show/generalize results using one framework?

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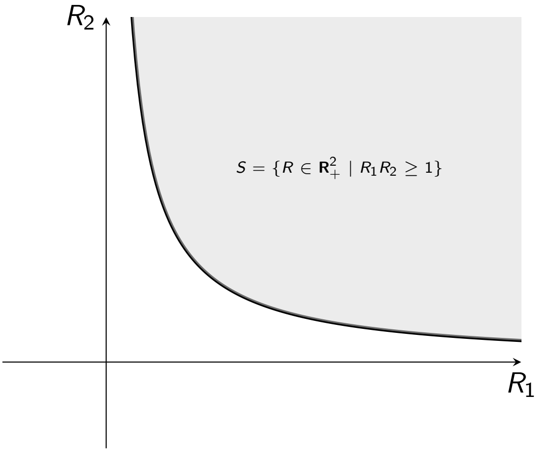
A basic definition

- A *constant function market maker* is defined by a set S
- The *reachable reserve set* $S \subseteq \mathbf{R}^n_+$ satisfies two properties:
  1. The set S is nonempty, closed, and convex
  2. The set is *upward closed*: if $R \in S$ and $R' \geq R$ then $R' \in S$

Pictures

For example Uniswap v1/v2



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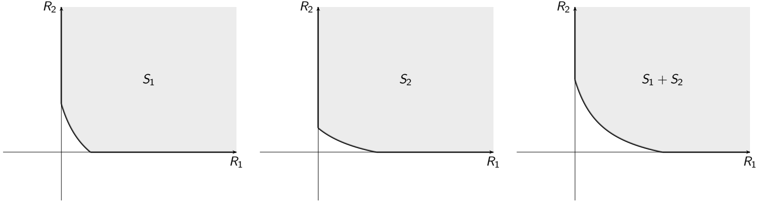
- Equivalently, can move to new reserves $R' \in S$ by paying Δ
- Includes original definition

$$S = \{R \mid \varphi(R) \geq k\}$$

Immediate consequences

- Reachable sets (therefore CFMMs) are closed under:
  1. Set (Minkowski) addition
  2. Scaling
  3. Intersection
  4. Nonnegative matrix multiplication

- Example of addition:



The trading function

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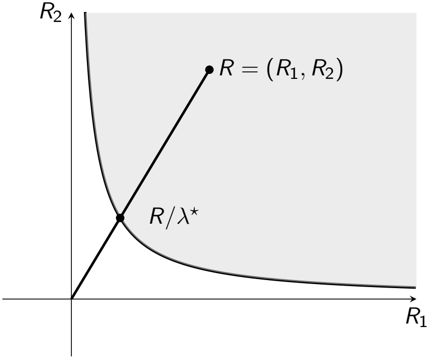

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- Canonical function is homogeneous, concave, nondecreasing
- Also unique(!) up to scaling
- Uniswap canonical trading function: $\varphi(R) = \sqrt{R_1 R_2}$

The liquidity cone

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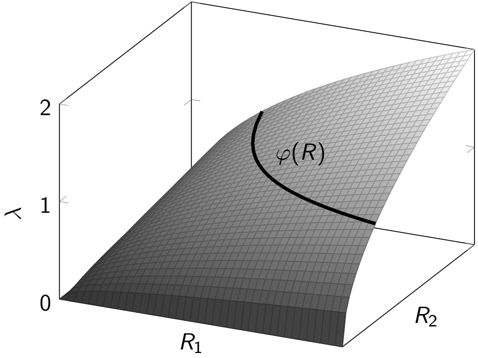


Liquidity cone equivalence

- Liquidity cone (from before)

$$K = \mathbf{cl}\{(R, \lambda) \mid R/\lambda \in S,\ \lambda > 0\}$$

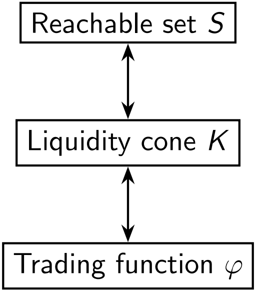
- The liquidity cone K 'contains' the reachable set S

$$S = \{R \mid (R, 1) \in K\}$$

- Cone K also gives canonical trading function

$$\varphi(R) = \sup\{\lambda \mid (R, \lambda) \in K\}$$

Where are we so far?



Outline

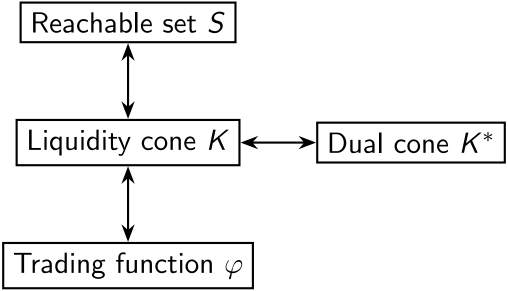
Dual cone

- A closed convex cone K has a convex *dual cone* K^*:

$$K^* = \{y \mid x^T y \geq 0 \text{ for all } x \in K\}$$

- This dual cone is another representation of the original cone
- Called the *dual* since $(K^*)^* = K$

A new challenger has arrived!



Dual cone for liquidity cone

- What is the dual of the liquidity cone?

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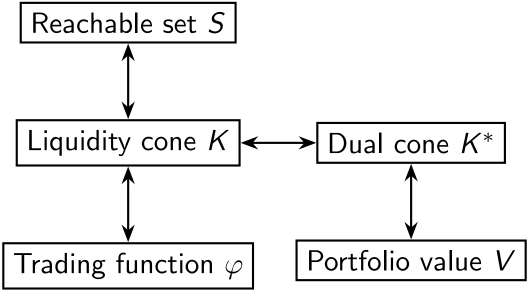
- For those familiar: looks a lot like the portfolio value function!

$$\begin{aligned} V(c) &= \inf\{c^T R \mid R \in S\} \\ &= \sup\{-\eta \mid (c, \eta) \in K^*\} \end{aligned}$$

The last domino



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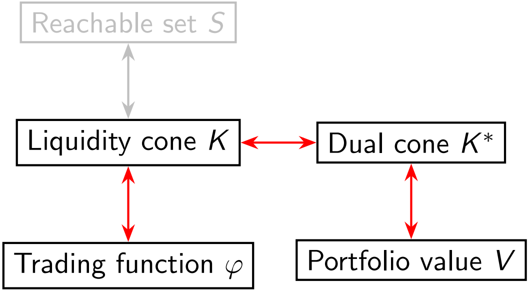

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- And backwards:

$$\varphi(R) = \inf_{c>0} \left(\frac{c^T R}{V(c)} \right)$$

- A more symmetric form than known previously

Operations

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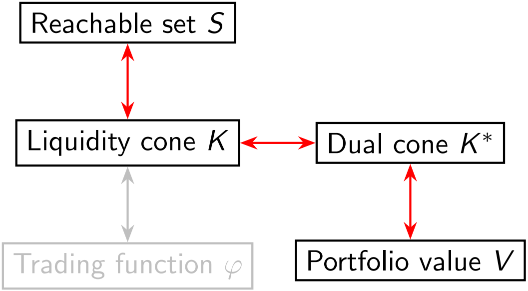
- Can we 'follow arrows' to find what they correspond to?



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- The 'fancy' statement:

  There is an operation-preserving isomorphism $S \leftrightarrow V$

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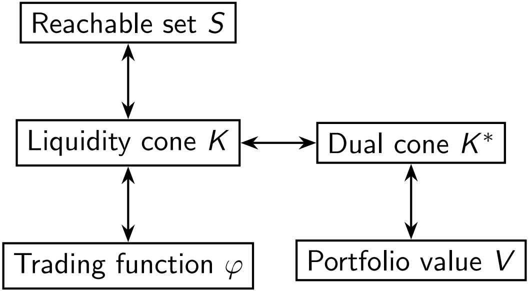
One more time



Paper has many more results

- Simple equivalences for path-independence
- General single-trade case
- Equivalence of routing and CFMM aggregation
- Connections to prediction markets (originally from Frongillo, Papireddygari, and Waggoner)
- Among many more!

Thanks!