Constant Function Market Makers

Pushing Uniswap & friends to do more with lower fees

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Outline

Trading assets

- Often, we need a way of trading assets
- (Normally) easy: make an offer to buy or sell
- In the traditional setting, this led to order books
  - Buyers post ‘bids’ of *maximum price* they are willing to buy for
  - Sellers post ‘asks’ of *minimum price* they are willing to sell for
  - Trades occur when a buyer is willing to pay more than minimum ask price

Disadvantages

- A trusted party keeps a record of outstanding bids and asks
  Linear space requirement
- When the highest bidder bids more than the lowest asker [...]
  Price may update slowly, esp. with a small number of agents
- Exchanges usually add subsidies to ensure there is liquidity

Alternatives to the tyranny of the order book

- Question: can we replace human market makers with algorithmic market makers?
- Yes! — Automated Market Makers (AMMs)
  - Bounded loss: can give explicit bounds on the maximum loss
  - Liquidity sensitivity: fixed-size trade moves prices less in thick markets
  - Small storage requirement: accepting or rejecting trade depends only on the current reserves

Automated Market Makers

Savage '71, Hanson '02

- Idea: use a (simple) formula to determine asset price
- *Liquidity providers* pool their assets (say A and B) into *reserves*
- Price set too low: agents purchase reserves at current price
- Price set too high: agents sell to reserves at current price

Automated Market Makers

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- Price set too low: agents purchase reserves at current price
- Price set too high: agents sell to reserves at current price
- Using this idea, set price based on assets remaining in reserves
- *e.g.*, if too much of asset A remains, compared to asset B, decrease the price of A

Automated Market Maker examples

- Simplest example: fixed asset price at all reserve amounts
  i.e., a flat line

- Another example: reported price is ratio of two asset reserves
  This curve is Uniswap!

A brief history of AMMs

- Savage '71, Hanson '02: Logarithmic Market Scoring Rules can be used for AMMs in prediction markets
- Chen, Pennock '07: axiomatic formulations exist for AMMs for more markets
- Othman, Sandholm '10: liquidity-sensitive AMMs exist that don't have unbounded loss
- Othman, Sandholm '11: constant utility AMMs for prediction markets
- Buterin, Koppelmann, '16: extension of AMMs to constant product markets, applications in exchanges

Outline

Uniswap (and constant product markets)

- Constant product markets (*e.g.*, Uniswap) is the family of curves whose reserves R_α, R_β must always satisfy:

$$R_\alpha R_\beta = k,$$

  for some constant k (no fees), we will call this the *trading function*

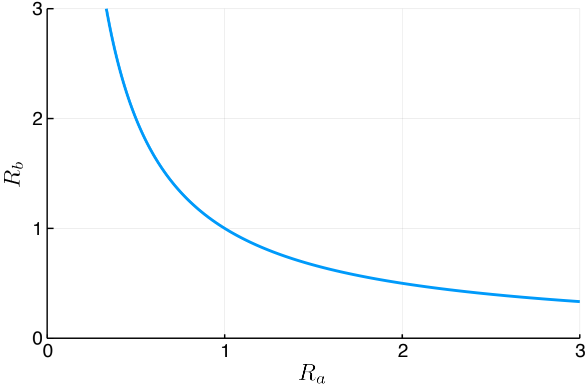
- In this case, we will assume that α and β are coins, though they can be any asset

- To satisfy this equation, the marginal price of asset β with respect to α is always

$$m_u = \frac{R_\beta}{R_\alpha}$$

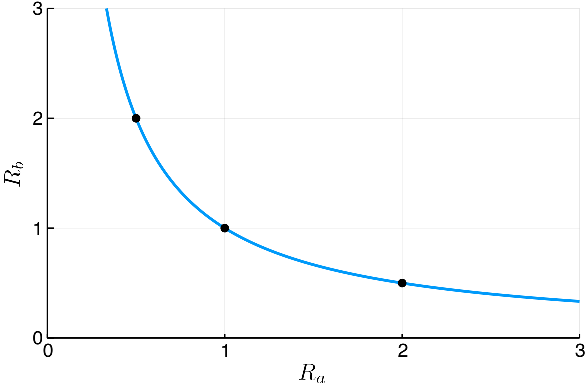
Uniswap market function

- If we plot the set of reserves that Uniswap can have, we get a hyperbola



Uniswap market function

- So, traders can exchange coin α and β with the reserves, so long as the resulting reserves remain on the curve



Analysis of market functions

- Uniswap is easy to analyze: trading function is simple
- What about more complicated cases? (*e.g.*, Balancer, Curve, *etc.*)
- While we could work out derivatives for prices, how do LP returns work? Are they good or bad for arbitrageurs?
- Additionally, many trading functions give the same trades!

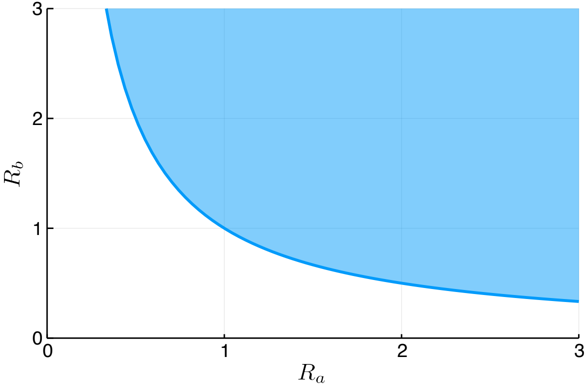
Outline

Filling in the graph

- Weird idea: what if we 'fill in' the graph?
- In other words, what if reserves above and to the right of the graph were also possible?

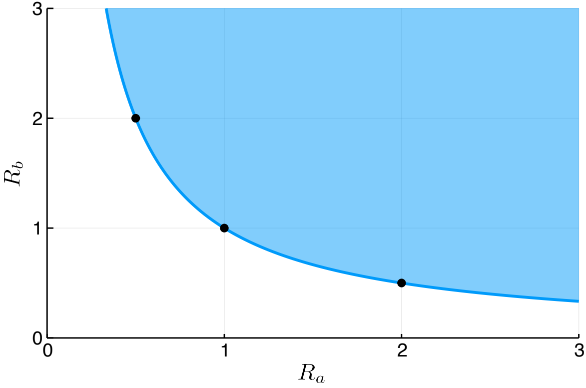
Uniswap trading set

- The new graph (or set) will look like



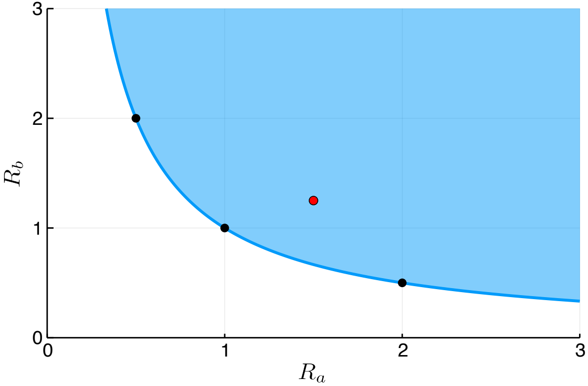
Uniswap trading set

- Since we've only added points, all of the reserves that were feasible before are still feasible



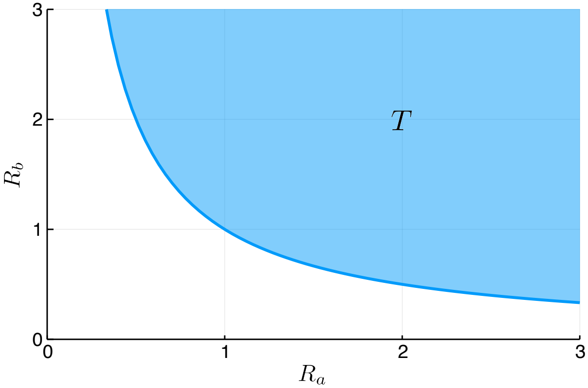
Uniswap trading set

- But, no rational trader will ever pick a point *inside* of the set!



Uniswap trading set

- We will call this set the *trading set*, $T \subseteq \mathbf{R}_+^2$, for short



The trading set

- The trading set T is often easier to deal with than the actual trading function and is the same for rational agents
- In particular, many trading functions yield the same set!
- Yet, the trading set is convex in practical scenarios (and all known CFMMs)
- Convex $\implies$ (a) easy to optimize and (b) easy to analyze

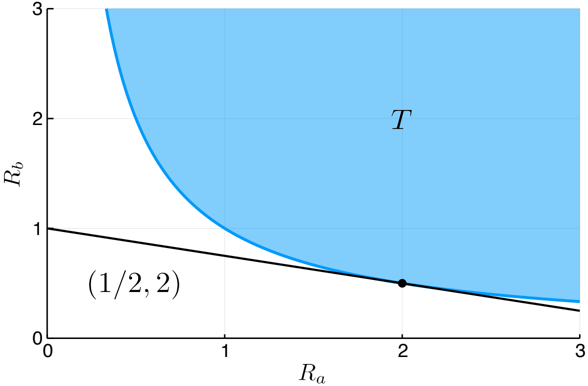
Outline

Constant function market makers

- Surprisingly, all CFMMs of this form have the same properties
- Marginal prices are simple to compute
- It is immediate that arbitrage is (computationally) easy, implying that prices match external ones
- Can compute liquidity provider returns, often in closed form
- All of this follows from basic convex analysis!

Marginal price

- Marginal price of a CFMM at given reserves is proportional to the *supporting hyperplane* of the set T at these reserves



Liquidity provider portfolio value

- The liquidity provider portfolio value is exactly given by

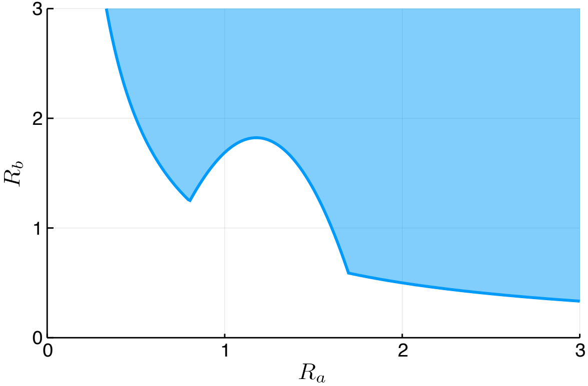
$$\inf_{r \in T} p^T r,$$

where p is the vector of prices with $p_i \geq 0$ the price of coin i

- This is the negative of the *support function* of the set T
- Almost always easy to evaluate for convex sets T
- First way of analytically computing liquidity provider returns of complicated CFMMs (*e.g.*, Balancer with n coins)

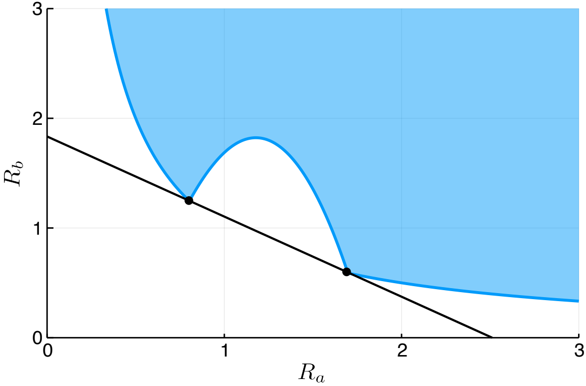
Why convexity?

- Sure, it's easy, but why?



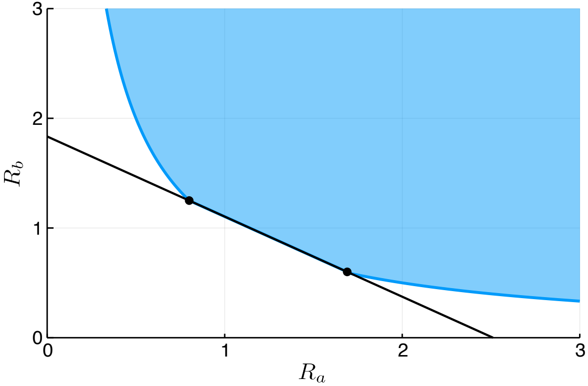
Why convexity?

- Note that arbitrageurs won't care! Marginal price is the same



Why convexity?

- So we might as well fill it in



Extensions and questions

- We can extend this to include fees (*path deficiency*), but is generally more complicated since T changes after each trade
- Of course, all of this holds for n coins, not just two
- Questions still remain... manipulation price? Lower bounds?

CFMM properties (summary)

- Main idea: recast all numerical properties of CFMMs as geometric properties of convex sets
- Can use this to talk about marginal prices, the arbitrage problem, liquidity provider returns, etc
- Second implication: it is possible to optimize over the set of CFMMs!
- We can make specific CFMMs for specific applications, including fees, curve shapes, etc.
- And many more possibilities

Outline

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